

ON THE RANK OF THE CHOW GROUP AND GROTHENDIECK GROUP OF AN ALGEBRAIC VARIETY

HELGE MAAKESTAD

ABSTRACT. The Chow group was introduced in the 1930's but still there are few general results on the rank of the Chow group of a smooth algebraic variety with rational or finite coefficients. Mumford proved in [12] that for a complex projective surface S with $H^0(S, \Omega_S^2) \neq 0$ it follows $\mathrm{CH}^*(S)$ is infinite dimensional in a generalized sense. Quite recently Schoen gave the first example of a complex projective manifold X and a prime number l where the Chow group $\mathrm{CH}^*(X) \otimes \mathbb{F}_l$ was infinite dimensional. Totaro proved in 2015 that for a very general principally polarized complex abelian 3-fold X , the Chow group $\mathrm{CH}^2(X) \otimes \mathbb{F}_l$ is infinite for all prime numbers l . In this note we prove that for any smooth complex algebraic variety X with an open affine subset $U = \mathrm{Spec}(A)$ where A/\mathbb{C} satisfies the PBW-property and $H_{\mathrm{sing}}^2(U, \mathbb{C}) \neq 0$, it follows the Chow group $\mathrm{CH}^*(X) \otimes k$ is uncountably infinite dimensional over k , where $k = \mathbb{Q}$ is the field of rational numbers or $k = \mathbb{F}_l$ for any prime number l . Here $H_{\mathrm{sing}}^2(U, \mathbb{C})$ is singular cohomology of U with complex coefficients. To prove this we construct a sub ring $\overline{K}_0(A)$ of the Grothendieck ring $K_0(A)$ where A is a finitely generated and regular algebra over the complex numbers. The ring $\overline{K}_0(A)$ is uncountably infinitely generated as abelian group when A/\mathbb{C} satisfies the PBW-property and $H_{\mathrm{sing}}^2(\mathrm{Spec}(A), \mathbb{C}) \neq 0$. Similar theorems may be proved using the techniques in this paper for algebraic varieties over the real numbers. We also study an affine version of a conjecture of Atiyah from 1957 on the existence of algebraic vector bundles with no flat algebraic connections. We give explicit examples of algebraic vector bundles on complex affine algebraic manifolds with no flat algebraic connection. We also discuss the Beauville conjecture on injectivity of the cycle map for divisors on projective hyper-Kähler manifolds and a possible relationship to the Bloch-Beilinson conjectures on algebraic cycles.

CONTENTS

1. Introduction	2
2. Some general results on universal enveloping algebras	4
3. The canonical sub-ring of $K_0(A)$ of type one	8
4. Some families of classes in the Grothendieck group of connections	16
5. On the rank of the Chow group of an algebraic variety	19
References	25

Date: December 2015.

1991 *Mathematics Subject Classification.* 14C25, 14C15, 14F40, 19A15.

Key words and phrases. connection, curvature, Chern character, Grothendieck group, Picard group, Chow group, rank.

1. INTRODUCTION

The study of algebraic cycles and the Chow group $\mathrm{CH}^*(X)$ of an algebraic variety X was introduced in the 1930's but still there are few general results on the rank and the existence of non-trivial elements in $\mathrm{CH}^*(X)$. The Chow group $\mathrm{CH}^*(X)$ has the reputation for being difficult to calculate in general. One wants to give for a large class of varieties X an explicit construction of non-trivial classes c in $\mathrm{CH}^*(X)$. The main aim of the paper is to solve this problem and to initiate a study of non-trivial sub-rings of the Grothendieck ring $K_0(X)$ and Chow ring $\mathrm{CH}^*(X)$ of an algebraic variety X over an arbitrary field. The aim is also to introduce general techniques for the calculation of lower bounds for ranks of Chow groups, Grothendieck groups and Picard groups of algebraic varieties.

We give a general construction of a non-trivial sub-ring $\overline{K}_0(X)$ of the Grothendieck ring $K_0(X)$ where $X = \mathrm{Spec}(A)$ and A is a commutative ring over a field of characteristic zero. The sub-ring $\overline{K}_0(X)$ is then applied to study similar questions for the Chow ring $\mathrm{CH}^*(X)$. The techniques introduced in the paper are general enough to give results on the rank of Chow groups, Grothendieck groups and Picard groups of algebraic varieties over arbitrary fields. We give results for Chow groups, Grothendieck groups and Picard groups of smooth varieties over the real and complex numbers (see Theorem 3.9, 4.2, 5.11, Corollary 4.5, 5.18 and Example 5.7). We also discuss a possible relationship to the Beauville conjecture on injectivity of the cycle map for divisors and to the Bloch-Beilinson conjectures on algebraic cycles.

Mumford proved in [12] that for a complex projective surface S with $H^0(S, \Omega_S^2) \neq 0$ it follows $\mathrm{CH}^*(S)$ is infinite dimensional in a generalized sense. He also proved that the Grothendieck group of coherent sheaves $K_0(S) \otimes \mathbb{Q}$ on S was infinite dimensional. Recent results of Schoen and Totaro suggests that the Chow ring $\mathrm{CH}^*(X) \otimes \mathbb{F}_l$ for l a prime is infinite dimensional in many cases (see [14] and [18]) and the main aim of this paper is to give a large class of examples where this holds.

In a recent paper (see [10]) I define for any affine algebraic manifold $X = \mathrm{Spec}(A)$ a *characteristic ring*

$$\mathrm{Char}(\mathrm{Der}_{\mathbb{C}}(A)) \subseteq H_{\mathrm{sing}}^{2*}(X_{\mathbb{C}}, \mathbb{C})$$

where $H_{\mathrm{sing}}^{2*}(X_{\mathbb{C}}, \mathbb{C})$ is the even part of singular cohomology of $X_{\mathbb{C}}$ with complex coefficients, with the property that $\mathrm{Char}(\mathrm{Der}_{\mathbb{C}}(A))$ is non-trivial if and only if $H_{\mathrm{sing}}^2(X_{\mathbb{C}}, \mathbb{C}) \neq 0$. When A/\mathbb{C} satisfies the *PBW-property* there is a surjective map of rings

$$\mathrm{Ch}_{\mathbb{Q}} : \overline{K}_0(\mathrm{Der}_{\mathbb{C}}(A)) \otimes \mathbb{Q} \rightarrow \mathrm{Char}(\mathrm{Der}_{\mathbb{C}}(A))$$

where

$$\overline{K}_0(\mathrm{Der}_{\mathbb{C}}(A)) \subseteq K_0(\mathrm{Der}_{\mathbb{C}}(A))$$

is a sub-ring of the Grothendieck ring $K_0(\mathrm{Der}_{\mathbb{C}}(A))$ of $\mathrm{Der}_{\mathbb{C}}(A)$ -connections. Hence if $H_{\mathrm{sing}}^2(X_{\mathbb{C}}, \mathbb{C}) \neq 0$ it follows $\overline{K}_0(\mathrm{Der}_{\mathbb{C}}(A))$ is a non-trivial sub-ring of $K_0(\mathrm{Der}_{\mathbb{C}}(A))$. This construction gives rise to a non-trivial subring

$$\overline{K}_0(A) \subseteq K_0(A)$$

of the Grothendieck ring $K_0(A)$ of finitely generated and projective A -modules - the *canonical sub-ring of $K_0(A)$ of type one*. In Theorem 3.9 the following result is proved:

Theorem 1.1. *Let A be a finitely generated and regular algebra over the complex numbers such that A/\mathbb{C} satisfies the PBW-property, and $H_{\mathrm{sing}}^2(\mathrm{Spec}(A)_{\mathbb{C}}, \mathbb{C}) \neq 0$.*

It follows $\overline{K}_0(\text{Spec}(A)) \otimes k$ has uncountable infinite dimension over k when $k = \mathbb{Q}$ or $k = \mathbb{F}_l$ where l is any prime.

Since there is an inclusion of rings $\overline{K}_0(A) \subseteq K_0(A)$ it follows the same result holds for the Grothendieck ring $K_0(A)$ of A . A similar result holds for $K_0(X)$ where X is an algebraic variety of finite type over the complex numbers containing an open affine subscheme $U = \text{Spec}(A)$ where A/\mathbb{C} satisfies the PBW-property and $H_{\text{sing}}^2(U, \mathbb{C}) \neq 0$.

The Grothendieck ring $K_0(A)$ is not a vector space over the complex numbers, but by Example 3.15 it follows the sub-ring $\overline{K}_0(A)$ has the cardinality of a finite dimensional complex vector space. There is an exact sequence of rings

$$0 \rightarrow \ker(\overline{Ch}_{\mathbb{Q}}) \rightarrow \overline{K}_0(A) \otimes \mathbb{Q} \rightarrow \text{Char}(\text{Der}_{\mathbb{C}}(A)) \rightarrow 0$$

where $\text{Char}(\text{Der}_{\mathbb{C}}(A))$ has the cardinality of a finite dimensional complex vector space.

Since the Chow group $\text{CH}^*(X)$ is the associated graded group of $K_0(A)$ with respect to the γ -filtration it follows $\text{CH}^*(X)$ is non-trivial. One may ask if we may use the characteristic ring $\text{Char}(\text{Der}_{\mathbb{C}}(A))$ and the gamma filtration to construct non-trivial subrings of the Chow ring $\text{CH}^*(X)$ solving the problem mentioned above, and the aim of this paper is to initiate such a study.

In Section 4 of the paper I construct the following: Let L be a Lie-Rinehart algebra which is finitely generated and projective of rank l as A -module and satisfies the PBW-property. For any pair of integers $k, i \geq 1$, I construct a set theoretic section

$$\psi^{k,i} : H^2(L, A) \rightarrow \overline{K}_0(L)$$

of the first Chern class map $c_1 : K_0(L) \rightarrow H^2(L, A)$. In Theorem 4.2 I prove various general properties of the map $\psi^{k,i}$.

When A is a finitely generated and regular algebra over the complex numbers, $L = \text{Der}_{\mathbb{C}}(A)$ and $H_{\text{sing}}^2(\text{Spec}(A), \mathbb{C}) \neq 0$ I prove for any pair of integers $k, i \geq 1$ the existence of a map of sets

$$\psi^{k,i} : H_{\text{sing}}^2(X, \mathbb{C}) \rightarrow \overline{K}_0(A).$$

In Corollary 4.5 the following is proved:

Corollary 1.2. *Assume A/\mathbb{C} satisfy the PBW-property and $H_{\text{sing}}^2(X, \mathbb{C}) \neq 0$. The following holds:*

$$(1.2.1) \quad c_1(\psi^{k,i}(c)) = c \in H_{\text{sing}}^2(X, \mathbb{C})$$

$$(1.2.2) \quad c \neq 0 \Rightarrow \psi^{k,i}(c) \neq 0 \text{ in } \overline{K}_0(A)$$

$$(1.2.3) \quad k \geq 1, i \neq j \geq 1 \Rightarrow \psi^{k,i}(c) \neq \psi^{k,j}(c) \text{ in } \overline{K}_0(A)$$

$$(1.2.4) \quad c \neq c' \Rightarrow \psi^{k,i}(c) \neq \psi^{k,i}(c') \text{ in } \overline{K}_0(A)$$

Hence the canonical sub-ring $\overline{K}_0(A)$ of type one contains large families of non-trivial classes parametrized by $H_{\text{sing}}^2(X, \mathbb{C})$ which is a finite dimensional complex vector space. When we vary the integers k, i we get inequivalent families. For all k, i the map

$$\psi^{k,i} : H_{\text{sing}}^2(X, \mathbb{C}) \rightarrow \overline{K}_0(A)$$

is an injection of sets. The group $\overline{K}_0(A)$ is not a vector space over the complex numbers but still it contains for every pair of integers $k, i \geq 1$ different copies of

the vector space $H_{sing}^2(X, \mathbb{C})$ via the map $\psi^{k,i}$. Hence the sub-ring $\overline{K_0}(A)$ is large in general.

In the final section of the paper I generalize a Theorem from an earlier paper on the subject (see [10]) and construct a class of finitely generated projective modules with connection with prescribed curvature (see Theorem 5.2). I use this construction to prove the following Theorem (see Theorem 5.11):

Theorem 1.3. *Let $X = \text{Spec}(A)$ where A is a regular algebra of finite type over the complex numbers. Assume A/\mathbb{C} satisfies the PBW-property and $H_{sing}^2(X_{\mathbb{C}}, \mathbb{C}) \neq 0$. It follows $\text{Pic}(A) \otimes k$ has uncountable dimension over k for $k = \mathbb{Q}$ or $k = \mathbb{F}_l$ for any prime l .*

A similar result holds for $\text{Pic}(X)$ where X is an algebraic variety of finite type over \mathbb{C} containing an open affine subscheme $U = \text{Spec}(A)$ where A/\mathbb{C} satisfies the PBW-property and $H_{sing}^2(U, \mathbb{C}) \neq 0$. A similar result holds for the Chow ring $\text{CH}^*(X)$ (see Corollary 5.18).

The *Atiyah conjecture* from [1] states that a holomorphic vector bundle on a complex projective manifold with a holomorphic connection always has a flat holomorphic connection. It is well known a holomorphic projective manifold X over the complex numbers is algebraic and a holomorphic vector bundle E on X is algebraic. Moreover a holomorphic connection

$$\nabla : E \rightarrow E \otimes \Omega_X^1$$

is algebraic. In the affine algebraic situation one may ask the following: Let $Y = \text{Spec}(A)$ be a complex affine algebraic manifold and let

$$\nabla : W \rightarrow W \otimes \Omega_Y^1$$

be an algebraic connection. Does this imply W has a flat algebraic connection? The answer to this is negative and we prove this by giving explicit examples of finitely generated projective A -modules with no flat algebraic connections (see Corollary 5.4).

We give examples indicating that the Chern class map

$$c_1 : \text{Pic}^L(A) \rightarrow H^2(L, A)$$

has non trivial kernel. We construct in Example 5.7 a large class of invertible modules $\omega \in \text{Pic}^L(A)$ with $c_1(\omega) = 0$. The invertible module ω is non-trivial in general. The *Beauville conjecture* says that for a projective hyper-Kaehler manifold X the cycle class map

$$(1.3.1) \quad \gamma : \text{CH}^*(X) \rightarrow H_{sing}^*(X, \mathbb{C})$$

is injective on the subalgebra of $\text{CH}^*(X)$ generated by divisors. In the algebraic case the invertible modules ω constructed in Example 5.7 globalize. It might be this construction can be useful in the study of the map 1.3.1 and the Beauville conjecture.

2. SOME GENERAL RESULTS ON UNIVERSAL ENVELOPING ALGEBRAS

Let in this section $R \rightarrow A$ be an arbitrary map of commutative unital rings and let $\alpha : L \rightarrow \text{Der}_R(A)$ be an arbitrary Lie-Rinehart algebra. Let $f \in Z^2(L, A)$ be 2-cocyle for L and let $U(A, L, f)$ be the generalized universal enveloping algebra of

L with respect to f as defined in [10] Definition 3.1. Let $U(A, L) := U(A, L, 0)$. It follows $U(A, L)$ is the classical enveloping algebra of Rinehart. It has the property there is a one to one correspondence between left $U(A, L)$ -modules and flat L -connections.

Lemma 2.1. *Let $A = U_0 \subseteq U_1 \subseteq \cdots \subseteq U_k \subseteq \cdots$ be a filtered associative unital algebra with the property that for $x \in U_i, y \in U_j$ it follows $xy - yx \in U_{i+j-1}$. It follows the associated graded algebra $Gr(U, U_i)$ is a commutative algebra. Moreover the canonical map*

$$\rho : \text{Sym}_A^*(U_1/U_0) \rightarrow Gr(U, U_i)$$

is a map of graded algebras.

Proof. The proof is clear. \square

Lemma 2.2. *Assume L is a projective A -module of rank l . Let f be an element in A where*

$$L_f \cong A_f \{x_1, \dots, x_l\}$$

as left A_f -module. Assume the canonical map

$$\text{Sym}_A(L) \cong Gr(U(A, L))$$

is an isomorphism. It follows there is an isomorphism

$$U(A, L)_f \cong A_f \{x_1^{p_1} \cdots x_l^{p_l} : p_i \geq 0\}$$

as left A -modules.

Proof. The associative algebra $U(A, L)$ is functorial in A and L hence there is an isomorphism

$$U(A, L)_f \cong U(A_f, L_f)$$

of left A -modules. The isomorphism $\text{Sym}_A(L) \cong Gr(U(A, L))$ implies there is an isomorphism

$$\text{Sym}_{A_f}(L_f) \cong Gr(U(A_f, L_f))$$

and the result follows. \square

Proposition 2.3. *Let $\alpha : L \rightarrow \text{Der}_R(A)$ be a Lie-Rinehart algebra and let $f, g \in Z^2(L, A)$. It follows there is a canonical isomorphism*

$$U(A, L, f) \cong U(A, L, g)$$

of left A -modules. The following holds: There is an isomorphism of graded A -algebras

$$\text{Sym}_A^*(L) \cong Gr(U(A, L))$$

if and only if there is an isomorphism of graded algebras

$$\text{Sym}_A^*(L) \cong Gr(U(A, L, f))$$

for all $f \in Z^2(L, A)$.

Proof. By definition there is an isomorphism of left A -modules

$$U(A, L, f) \cong U(A, L, g)$$

hence the first claim is clear. We prove the second claim: Assume there is an isomorphism

$$\rho : \text{Sym}_A^*(L) \cong Gr(U(A, L))$$

of graded A -algebras and let $f \in Z^2(L, A)$. The canonical isomorphism $U(A, L) \cong U(A, L, f)$ of filtered left A -modules induce an isomorphism of graded left A -modules

$$\psi : Gr(U(A, L)) \cong Gr(U(A, L, f))$$

We get an isomorphism

$$\psi \circ \rho : \text{Sym}_A^*(L) \rightarrow Gr(U(A, L, f))$$

which is an isomorphism of graded A -algebras. Hence one implication is proved. The other implication is obvious and the Proposition follows. \square

Proposition 2.3 gives a simplified proof of Theorem 3.7 in [10].

Definition 2.4. Let $\alpha : L \rightarrow \text{Der}_R(A)$ be a Lie-Rinehart algebra. If the canonical map

$$\rho : \text{Sym}_A^*(L) \rightarrow Gr(U(A, L))$$

is an isomorphism we say that L satisfies the PBW-property. We say the pair A/R satisfies the PBW-property if the Lie-Rinehart algebra $\text{Der}_R(A)$ satisfies the PBW-property.

Let $f \in Z^2(L, A)$ and let $L(f) := Az \oplus L$ in the canonical extension

$$Az \rightarrow Az \oplus L \rightarrow L$$

of Lie-Rinehart algebras. Let

$$p : L(f) \rightarrow U(A, L(f))$$

be the canonical map and let

$$I_f := \{p(z) - 1\}$$

be the 2-sided ideal in $U(A, L(f))$ generated by the element $p(z) - 1$.

Definition 2.5. Let $\tilde{U}(A, L(f)) := U(A, L(f))/I_f$.

There is the canonical projection map

$$q : U(A, L(f)) \rightarrow \tilde{U}(A, L(f)).$$

Let $\tilde{U}^k(A, L(f)) := q(U^k(A, L(f)))$ where $U^k(A, L(f))$ is the descending filtration of $U(A, L(f))$ as defined in [10], Section 3.

We get a descending filtration

$$\dots \tilde{U}^k(A, L(f)) \subseteq \dots \subseteq \tilde{U}^2(A, L(f)) \subseteq \tilde{U}^1(A, L(f)) = \tilde{U}(A, L(f)).$$

Definition 2.6. Let $\alpha : L \rightarrow \text{Der}_R(A)$ be a Lie-Rinehart algebra. Let $k, i \geq 1$ be integers and let $f \in Z^2(L, A)$. Let

$$\tilde{V}^{k,i}(A, L(f)) := \tilde{U}^k(A, L(f))/\tilde{U}^{k+i}(A, L(f)).$$

It follows $\tilde{V}^{k,i}(A, L(f))$ is a left $\tilde{U}(A, L(f))$ -module.

Proposition 2.7. Let $\alpha : L \rightarrow \text{Der}_R(A)$ be a Lie-Rinehart algebra and let $f \in Z^2(L, A)$. Assume L is a projective A -module of rank l . There is an equivalence of categories between the category of left $\tilde{U}(A, L(f))$ -modules and the category of L -connections of curvature type f . If $L(f)$ satisfies the PBW-property it follows $\tilde{V}^{k,i}(A, L(f))$ is a projective A -module of rank $r = \binom{l+k+i-1}{l} - \binom{l+k-1}{l}$. The rank of $\tilde{V}^{k,i}(A, L(f))$ is independent of choice of $f \in Z^2(L, A)$.

Proof. By definition there is an equivalence of categories between the category of left $\tilde{U}(A, L(f))$ -modules and the category of flat $L(f)$ -connection ∇ with $\nabla(z) = Id$. By [10], Lemma 4.17 it follows there is an equivalence of categories between the category of left $\tilde{U}(A, L(f))$ -modules and the category of L -connections of curvature type f and the first claim is proved.

We prove the second claim. Let $f \in A$ be an element where L trivialize. Hence there is an isomorphism

$$L_f \cong A_f \{x_1, \dots, x_l\}$$

of left A_f -modules. It follows there is an isomorphism

$$U(A_f, L_f) \cong A_f \{z^p x_1^{p_1} \dots x_l^{p_l} : z, p_i \geq 0\}$$

of left A_f -modules. It follows there is an isomorphism

$$\tilde{U}(A, L(f)) \cong A_f \{x_1^{p_1} \dots x_l^{p_l} : p_i \geq 0\}$$

of left A_f -modules. The second claim now follows. The third claim is clear and the Proposition is proved. \square

Assume in the following $\mathbb{Q} \subseteq R$ is the field of rational numbers. Let $\alpha : L \rightarrow \text{Der}_R(A)$ be a Lie-Rinehart algebra and let $\bar{f} = c \in H^2(L, A)$

Definition 2.8. Let $k, i \geq 1$ be integers. Let

$$\tilde{U}(A, L(c)) := \tilde{U}(A, L(f))$$

Let also

$$\tilde{V}^{k,i}(A, L(c)) := \tilde{U}^k(A, L(f)) / \tilde{U}^{k+i}(A, L(f)).$$

By functoriality it follows Definition 2.8 is well defined. We get for each cohomology class $c \in H^2(L, A)$ a filtered associative algebra $\tilde{U}(A, L(c))$ and a left $\tilde{U}(A, L(c))$ -module $\tilde{V}^{k,i}(A, L(c))$. Hence the cohomology group $H^2(L, A)$ parametrize for each pair of integers $k, i \geq 1$ a set of filtered associative algebras and modules.

Let $c = \bar{f} \in H^2(L, A)$.

Lemma 2.9. Assume $\alpha : L \rightarrow \text{Der}_R(A)$ satisfies the PBW-property. It follows the left A -module $\tilde{V}^{k,i}(A, L(c))$ is a finitely generated and projective A -module. of rank

$$r = \binom{l+k+i-1}{l} - \binom{l+k-1}{l}.$$

Proof. There is by [10], Theorem 3.3 a canonical isomorphism of filtered rings

$$\tilde{U}(A, L(f)) \cong U(A, L, f)$$

It follows there is an isomorphism of left A -modules

$$\tilde{V}^{k,i}(A, L(f)) \cong V^{k,i}(A, L, f)$$

where $V^{k,i}(A, L, f)$ is the module from [10], Definition 5.1. Since $U(A, L)$ satisfies the PBW-property it follows from Proposition 2.3 $U(A, L, f)$ satisfies the PBW-property. It follows from [10], Lemma 5.2 $\tilde{V}^{k,i}(A, L(f))$ is a finitely generated and projective A -module of rank

$$r = \binom{l+k+i-1}{l} - \binom{l+k-1}{l}.$$

The Lemma is proved. \square

Assume L satisfies the PBW-property and let $\bar{f} = c \in H^2(L, A)$. By Lemma 2.9 it follows for all integers $k, i \geq 1$ the module $\tilde{V}^{k,i}(A, L(c))$ is finitely generated and projective of rank r . Let for any $\bar{f} = c$, $\tilde{c} = \frac{1}{r}c \in H^2(L, A)$. Define the following family of A -modules $\phi^{k,i}$ parametrized by $H^2(L, A)$:

$$\phi^{k,i}(c) := \tilde{V}^{k,i}(A, L(\tilde{c})).$$

Theorem 2.10. *Assume R contains the field of rational numbers and assume $\alpha : L \rightarrow \text{Der}_R(A)$ is a Lie-Rinehart algebra which satisfies the PBW-property. Assume $c = \bar{f} \in H^2(L, A)$. The left A -module $\phi^{k,i}(c)$ is a finitely generated and projective A -module of rank $r = \binom{l+k+i-1}{l} - \binom{l+k-1}{l}$. There is a connection*

$$\nabla : L \rightarrow \text{End}_R(\phi^{k,i}(c))$$

of curvature type $\frac{1}{r}f$. The map $\phi^{k,i}$ is a set theoretic section of the first Chern class map. Hence

$$c_1(\phi^{k,i}(c)) = c$$

for all $c \in H^2(L, A)$.

Proof. The first and the second claim follows from Proposition 2.7 and Lemma 2.9.

We prove the last claim: We get

$$c_1(\phi^{k,i}(c)) = \overline{\text{tr}(R\nabla)}$$

where

$$\nabla : L \rightarrow \text{End}_R(\phi^{k,i}(c))$$

is a connection of curvature type $\frac{1}{r}f$. We get

$$c_1(\phi^{k,i}(c)) = \overline{\text{tr}\left(\frac{1}{r}fId\right)} = \overline{\frac{1}{r}f\text{tr}(Id)} = \bar{f} = c.$$

The Theorem is proved. \square

3. THE CANONICAL SUB-RING OF $K_0(A)$ OF TYPE ONE

Let in the following $R \rightarrow A$ be an arbitrary map of commutative unital rings where R contains a field of characteristic zero and let $\alpha : L \rightarrow \text{Der}_R(A)$ be a Lie-Rinehart algebra satisfying the PBW-property. In this section we prove some general results on the Chern character

$$Ch : K_0(L) \rightarrow H^*(L, A)$$

introduced in [11] and the characteristic ring $Char(L)$ introduced in [10]. In the case when L is finitely generated and projective as A -module there is by the results of [10] a subring

$$Char(L) \subseteq H^*(L, A).$$

It has the property that $Char(L)$ is non-trivial if and only if $0 \neq H^2(L, A)$. There is moreover a sub-ring

$$\overline{K}_0(L) \subseteq K_0(L)$$

of the Grothendieck ring $K_0(L)$ of finitely generated projective A -modules with an L -connection. The Chern character Ch induce a surjective map of rings

$$\overline{Ch}_{\mathbb{Q}} : \overline{K}_0(L) \otimes \mathbb{Q} \rightarrow Char(L).$$

Proposition 3.1. *Assume L is finitely generated and projective as A -module and satisfies the PBW-property. If $H^2(L, A) \neq 0$ it follows $K_0(L)$ is non-trivial.*

Proof. By [10] the following holds: If $H^2(L, A) \neq 0$ it follows $Char(L)$ is non-trivial. By Proposition 3.13 which is proved later in this section the map $\overline{Ch}_{\mathbb{Q}}$ is surjective. It follows $\overline{K}_0(L)$ is non-trivial. The Proposition follows since there is an inclusion of rings

$$\overline{K}_0(L) \subseteq K_0(L).$$

□

There is a canonical surjective map of rings

$$p : K_0(L) \rightarrow K_0(A)$$

defined by $p([E, \nabla]) = [E]$ where

$$\nabla : L \rightarrow \text{End}_R(E)$$

is an L -connection and E is a finitely generated and projective A -module.

Definition 3.2. Let $\overline{K}_0(A) := p(\overline{K}_0(L))$ be the image of $\overline{K}_0(L)$ under the map p . Let $\overline{K}_0(A)$ be the *canonical sub-ring of $K_0(A)$ of type one*.

Since p is a map of rings we get a sub-ring $\overline{K}_0(A) \subseteq K_0(A)$.

Assume in the following that $R = \mathbb{C}$ is the field of complex numbers and let A be a regular algebra of finite type over \mathbb{C} . Let $X = \text{Spec}(A)$ and let $X_{\mathbb{C}}$ be the underlying complex algebraic manifold in the strong topology. We get by the results of [10] A sub-ring

$$Char(\text{Der}_{\mathbb{C}}(A)) \subseteq H_{sing}^{2*}(X_{\mathbb{C}}, \mathbb{C})$$

where $H_{sing}^{2*}(X_{\mathbb{C}}, \mathbb{C})$ is the even part of singular cohomology of $X_{\mathbb{C}}$ with complex coefficients.

Corollary 3.3. *Assume A is a regular algebra of finite type over the complex numbers such that A/\mathbb{C} satisfies the PBW-property. Let $X = \text{Spec}(A)$. If $H_{sing}^2(X_{\mathbb{C}}, \mathbb{C}) \neq 0$ it follows $K_0(A)$ is non-trivial.*

Proof. The Corollary follows from Proposition 3.13 since the map \overline{Ch} is surjective. □

Let E, F be finite rank projective A -modules with projective bases $\{e_i, x_i\}$ and $\{u_j, y_j\}$ for $1 \leq i \leq m$ and $1 \leq j \leq n$ (see [10] for the definition of a projective basis).

Lemma 3.4. *The set $\{e_i \otimes u_j, x_i \otimes y_j\}$ with $1 \leq i \leq m$ and $1 \leq j \leq n$ is a projective basis for $E \otimes_A F$.*

Proof. Let $u \otimes v \in E \otimes_R F$. It follows

$$\begin{aligned} \sum_{i,j} x_i \otimes y_j (u \otimes v) e_i \otimes u_j &= \\ \sum_{i,j} x_i(u) e_i \otimes y_j(v) u_j &= \\ \left(\sum_i x_i(u) e_i \right) \otimes \left(\sum_j y_j(v) u_j \right) &= u \otimes v. \end{aligned}$$

The Lemma follows. □

The canonical map

$$\rho : E^* \otimes E \rightarrow \text{End}_A(E)$$

defined by

$$\rho(\phi \otimes e)(x) = \phi(x)e$$

is an isomorphism of A -modules. It has inverse given by

$$\rho^{-1}(\phi) = \sum_i x_i \otimes \phi(e_i).$$

There is a canonical trace map

$$\text{tr} : E^* \otimes_A E \rightarrow A$$

defined by

$$\text{tr}(\phi \otimes e) = \phi(e).$$

The trace of an endomorphism $\phi \in \text{End}_A(E)$ is defined as follows:

$$\text{tr}(\phi) = \sum_i x_i(\phi(e_i))$$

where $\{e_i, x_i\}$ is a projective basis for E . Given connections

$$\nabla : L \rightarrow \text{End}_R(E)$$

and

$$\nabla' : L \rightarrow \text{End}_R(F)$$

It follows we get a tensor product connection

$$\nabla \otimes \nabla' : L \rightarrow \text{End}_R(E \otimes_A F)$$

defined by

$$\nabla \otimes \nabla'(x)(u \otimes v) = \nabla(x)(u) \otimes v + u \otimes \nabla'(x)(v).$$

Proposition 3.5. *Assume E, F are finitely generated and projective A -modules with L -connections ∇, ∇' . The following holds:*

$$\text{tr}(R_{\nabla \otimes \nabla'}) = \text{rk}(F)\text{tr}(R_{\nabla}) + \text{rk}(E)\text{tr}(R_{\nabla'}).$$

Proof. For all $x \in L$ it follows $R_{\nabla \otimes \nabla'}(x) \in \text{End}_A(E \otimes_A F)$. We use ρ^{-1} and the results above to get the following element:

$$\sum_{i,j} x_i \otimes y_j \otimes R_{\nabla \otimes \nabla'}(e_i \otimes u_j) \in (E \otimes_A F)^* \otimes E \otimes_A F.$$

We get

$$\begin{aligned} \text{tr}(R_{\nabla \otimes \nabla'}) &= \\ \sum_{i,j} x_i \otimes y_j \otimes (R_{\nabla}(e_i) \otimes u_j + e_i \otimes R_{\nabla'}(u_j)) &= \\ \sum_{i,j} x_i(R_{\nabla}(e_i))y_j(u_j) + \sum_{i,j} x_i(e_i)y_j(R_{\nabla'}(u_j)) &= \\ (\sum_i x_i(e_i))(\sum_j y_j(R_{\nabla'}(u_j))) + (\sum_i x_i(R_{\nabla}(e_i)))(\sum_j y_j(u_j)) &= \\ \text{rk}(E)\text{tr}(R_{\nabla'}) + \text{tr}(R_{\nabla})\text{rk}(F), \end{aligned}$$

since $\sum_i x_i(e_i) = \text{rk}(E)$ when $\{e_i, x_i\}$ is a projective basis. The Proposition follows. \square

Corollary 3.6. *Let $(E, \nabla), (F, \nabla')$ be L -connections where E, F are finitely generated and projective as A -modules. It follows*

$$c_1(E \otimes_A F) = rk(F)c_1(E) + rk(E)c_1(F).$$

Assume E, F are invertible A -modules. It follows

$$c_1(E \otimes_A F) = c_1(E) + c_1(F).$$

Proof. The first claim follows from Proposition 3.5 since $c_1(E \otimes_A F) = \overline{tr(R_{\nabla \otimes \nabla'})}$ in $H^2(L, A)$. The second claim follows since $rk(E) = 1$ when E is an invertible A -module. \square

Let $\tilde{V}^{k,i}(A, L(f))$ be the A -module constructed from Definition 2.6. The following result follows from Theorem 2.10

Proposition 3.7. *Assume L is finitely generated and projective as A -module satisfying the PBW-property. Let R contain a field of characteristic zero or characteristic p where $p \geq rk(\tilde{V}^{k,i}(A, L(f)))$ for some integers $k, i \geq 1$. It follows the Chern-class map*

$$c_1 : K_0(L) \rightarrow H^2(L, A)$$

is a surjective map of abelian groups.

Proof. In characteristic zero the result is proved in Theorem 2.10. In characteristic $p > rk(\tilde{V}^{k,i}(A, L(f)))$ it follows from the proof of Theorem 2.10. \square

Hence in characteristic $p > rk(\tilde{V}^{k,i}(A, L(f)))$ we get a surjection of abelian groups

$$(3.7.1) \quad c_1 : K_0(L) \rightarrow H^2(L, A).$$

Example 3.8. *Ranks of Grothendieck groups of affine varieties in characteristic $p > 0$.*

The group $H^2(L, A)$ is calculated by the Lie-Rinehart complex and we get from 3.7.1 an estimate on the size of the Grothendieck group:

$$(3.8.1) \quad rk(K_0(L)) \geq rk(H^2(L, A)).$$

Hence we may use Proposition 3.7 to estimate the Grothendieck group for varieties in positive characteristic.

Theorem 3.9. *Let A be a finitely generated and regular algebra over the complex numbers such that A/\mathbb{C} satisfies the PBW-property. Let $X = \text{Spec}(A)$. If $H_{sing}^2(X_{\mathbb{C}}, \mathbb{C}) \neq 0$ it follows $\overline{K}_0(A) \otimes k$ is uncountably infinite dimensional over k where $k = \mathbb{Q}$ or $k = \mathbb{F}_l$ where l is any prime.*

Proof. There is by Theorem 2.10 a surjective map of abelian groups

$$p : \overline{K}_0(\text{Der}_{\mathbb{C}}(A)) \rightarrow \overline{K}_0(A)$$

giving a surjection of abelian groups

$$c_1 : \overline{K}_0(A) \rightarrow H_{sing}^2(X_{\mathbb{C}}, \mathbb{C}).$$

Let

$$\overline{K}_0(A) = T \oplus (\oplus_{i \in I} \mathbb{Z} e_i)$$

where T is the torsion subgroup of $\overline{K}_0(A)$ and $\oplus_{i \in I} \mathbb{Z}e_i$ is a free abelian group on the set I . We get an induced surjective map of abelian groups

$$c_1^{\mathbb{Q}} : \overline{K}_0(A) \otimes \mathbb{Q} \rightarrow H_{sing}^2(X_{\mathbb{C}}, \mathbb{C})$$

And $\overline{K}_0(A) \otimes \mathbb{Q} \cong \oplus_{i \in I} \mathbb{Q}e_i$. Since $c_1^{\mathbb{Q}}$ is surjective it follows the set I is uncountable and the result follows. \square

Corollary 3.10. *Let X be an algebraic variety of finite type over the complex numbers. Assume $U = \text{Spec}(A)$ is an open affine subscheme of X where A/\mathbb{C} satisfies the PBW-property and $H_{sing}^2(U, \mathbb{C}) \neq 0$. It follows $K_0(A) \otimes k$ is of uncountable infinite dimension over k where k is the field of rational numbers or the finite field \mathbb{F}_l where l is any prime.*

Proof. Let $Z = X - U$. There is an exact sequence of abelian groups

$$K_0(Z) \rightarrow K_0(X) \rightarrow K_0(U) \rightarrow 0.$$

The result follows from Theorem 3.9 since the sequence is right exact and the fact that the result holds for $K_0(A)$. \square

Corollary 3.11. *Assume $X = \text{Spec}(A)$ is smooth and that A/\mathbb{C} satisfies the PBE-property. Assume $H_{sing}^2(X_{\mathbb{C}}, \mathbb{C}) \neq 0$. It follows $K_0(A)$ has an uncountably infinite set of generators as abelian group.*

Proof. The claim follows from Corollary 3.9. \square

Let us recall the definition of the characteristic ring $\text{Char}(L)$ of a Lie-Rinehart algebra L from [10]. Assume in the following $\alpha : L \rightarrow \text{Der}_R(A)$ is a Lie-Rinehart algebra which is finitely generated and projective as A -module where A contains the field \mathbb{Q} of rational numbers. Recall the following map:

$$\exp : H^2(L, A) \rightarrow \oplus_{i \geq 0} H^{2i}(L, A)$$

defined by

$$\exp(x) = \sum_{i \geq 0} \frac{x^i}{i!}.$$

It follows $\exp(x+y) = \exp(x)\exp(y)$ for all $x, y \in H^2(L, A)$. Let

$$\text{Char}(L) = \left\{ \sum_i r_i \exp(x_i) : x_i \in H^2(L, A), r_i \in \mathbb{Q} \right\}.$$

Lemma 3.12. *The set $\text{Char}(L)$ is a sub-ring of $H^{2*}(L, A)$.*

Proof. Assume $u = \sum_i r_i \exp(x_i)$ and $v = \sum_j s_j \exp(y_j)$ are in $\text{Char}(L)$. It follows

$$\begin{aligned} uv &= \sum_{i,j} r_i s_j \exp(x_i) \exp(y_j) = \\ &= \sum_{i,j} r_i s_j \exp(x_i + y_j). \end{aligned}$$

It follows $uv \in \text{Char}(L)$ and the Lemma follows. \square

Recall the construction of the Chern-character

$$Ch : K_0(L) \rightarrow H^*(L, A)$$

from [11] and the sub-ring $\overline{K}_0(L) \subseteq K_0(L)$ from [10].

Proposition 3.13. *Assume L satisfies the PBW-property. It follows the Chern-character Ch induce a surjective map of rings*

$$\overline{Ch}_{\mathbb{Q}} : \overline{K}_0(L) \otimes \mathbb{Q} \rightarrow Char(L)$$

Proof. Let $[E, \nabla] \in \overline{K}_0(L)$ be an L -connection where E is a finitely generated and projective A -module and $R_{\nabla} = fId_E$ where $f \in Z^2(L, A)$. It follows

$$\begin{aligned} Ch([E, \nabla]) &= \\ \sum_{k \geq 0} \frac{tr(R_{\nabla}^k)}{k!} &= \\ 1 + rk(E) \sum_{k \geq 1} \frac{f^k}{k!} &= 1 - rk(E) + rk(E)exp(\overline{f}). \end{aligned}$$

Let $x = \overline{f} \in H^2(L, A)$. It follows

$$Ch([E, \nabla]) = (1 - rk(E)) + rk(E)exp(x) \in Char(L).$$

Assume $v = \sum_i r_i exp(x_i)$ with $r_i \in \mathbb{Q}$. It follows from Theorem 2.10 there is a connection (E_i, ∇_i) for every i with $R_{\nabla_i} = f_i Id_{E_i}$ and $\overline{f}_i = x_i$ in $H^2(L, A)$. Moreover E_i is a finitely generated and projective A -module. It follows

$$exp(x_i) = Ch\left(\frac{1}{rk(E_i)}([E_i, \nabla_i] - (rk(E_i) - 1))\right)$$

and

$$Ch\left(\sum_i \frac{r_i}{rk(E_i)}([E_i, \nabla_i] - (rk(E_i) - 1))\right) = \sum_i r_i exp(x_i) = v.$$

Hence $\overline{Ch}_{\mathbb{Q}}$ is surjective and the Proposition follows. \square

We sum up this in a Theorem.

Theorem 3.14. *Assume $\alpha : L \rightarrow \text{Der}_R(A)$ is a Lie-Rinehart algebra where L is finitely generated and projective as A -module satisfying the PBW-property. Assume moreover that R contains the field of rational numbers and that $H^2(L, A) \neq 0$. There is a surjective map of rings*

$$\overline{Ch}_{\mathbb{Q}} : \overline{K}_0(L) \otimes \mathbb{Q} \rightarrow Char(L).$$

The ring $Char(L)$ is non-trivial if and only if $H^2(L, A) \neq 0$. Moreover

$$\dim_{\mathbb{Q}}(Char(L)) = \dim_{\mathbb{Q}}(H^2(L, A)).$$

Proof. The claim on surjectivity follows from Proposition 3.13. The second claim is by definition. We prove the third claim: Let $\{x_i\}_{i \in I}$ be a basis for $H^2(L, A)$ as \mathbb{Q} -vector space. Define the following map

$$\phi : H^2(L, A) \rightarrow Char(L)$$

by

$$\phi\left(\sum_i r_i x_i\right) = \sum_i r_i exp(x_i).$$

One checks ϕ is an isomorphism of vector spaces over \mathbb{Q} and the Theorem follows. \square

Example 3.15. *Sub-algebras of the Grothendieck ring $K_0(A)$.*

Note: If A is a regular algebra of finite type over the complex numbers it follows from Theorem 3.14 that $\text{Char}(\text{Der}_{\mathbb{C}}(A))$ has the cardinality of a finite dimensional complex vector space: There is an isomorphism

$$H_{\text{sing}}^2(\text{Spec}(A), \mathbb{C}) \cong \text{Char}(\text{Der}_{\mathbb{C}}(A))$$

as vector spaces over \mathbb{Q} . Since \mathbb{C} has uncountable infinite dimension over \mathbb{Q} it follows $\text{Char}(\text{Der}_{\mathbb{C}}(A))$ has uncountably infinite dimension over \mathbb{Q} . Hence when A/\mathbb{C} satisfies the PBW-property it follows the subring

$$\overline{K}_0(A) \subseteq K_0(A)$$

sits in an exact sequence

$$0 \rightarrow \ker(\overline{Ch}_{\mathbb{Q}}) \rightarrow \overline{K}_0(A) \otimes \mathbb{Q} \rightarrow \text{Char}(\text{Der}_{\mathbb{C}}(A)) \rightarrow 0.$$

where

$$\overline{Ch}_{\mathbb{Q}} : \overline{K}_0(A) \otimes \mathbb{Q} \rightarrow \text{Char}(\text{Der}_{\mathbb{C}}(A))$$

is the map induced by Ch . The Grothendieck ring $K_0(A)$ is not a vector space over the complex numbers but still $K_0(A)$ contains a subalgebra $\overline{K}_0(A)$ with the cardinality of a finite dimensional complex vector space in the case when

$$H_{\text{sing}}^2(\text{Spec}(A)_{\mathbb{C}}, \mathbb{C}) \neq 0.$$

One may ask for a description of the ideal $\ker(\overline{Ch}_{\mathbb{Q}}) \subseteq \overline{K}_0(A) \otimes \mathbb{Q}$ to give a description of $\overline{K}_0(A)$ as a \mathbb{Q} -algebra.

We may view $K_0(A)$ as a module on $\overline{K}_0(A)$ and if $\overline{K}_0(A)$ is large in $K_0(A)$ it may be $K_0(A)$ is a finitely generated $\overline{K}_0(A)$ -module. Since $\overline{K}_0(A)$ has uncountably infinite rank as abelian group this might be the case. The sub ring $\overline{K}_0(A)$ contains by Theorem 3.14 (see [10]) projective modules E of arbitrary high rank as A -module. One wants to give a geometric construction of finitely generated and projective A -modules E_1, \dots, E_k and an equality

$$K_0(A) = \overline{K}_0(A)\{[E_1], \dots, [E_k]\}$$

as $\overline{K}_0(A)$ -modules.

Example 3.16. *Some explicit examples and Rineharts classical PBW-theorem.*

Let $A = \mathbb{C}[x, 1/x, y, 1/y]$ and $U = \text{Spec}(A) = \mathbb{C}^ \times \mathbb{C}^* \subseteq \mathbb{C}^2$. Let $Z = \mathbb{C}^2 - U$. We use the algebraic de Rham complex to calculate singular cohomology of U with complex coefficients. It follows $H_{\text{sing}}^2(U, \mathbb{C}) = \mathbb{C}$ is one dimensional. By localization it follows there is an exact sequence of abelian groups*

$$K_0(Z) \rightarrow K_0(\mathbb{C}^2) \rightarrow K_0(U) \rightarrow 0$$

and since $K_0(\mathbb{C}^2) = \mathbb{Z}$ it follows $K_0(U)$ is a quotient of \mathbb{Z} . Since $H_{\text{sing}}^2(U, \mathbb{C}) = \mathbb{C}$ it follows The Chern character

$$(3.16.1) \quad Ch \otimes \mathbb{Q} : K_0(U) \otimes \mathbb{Q} \rightarrow H_{\text{sing}}^2(U, \mathbb{C})$$

can not be surjective. It follows the pair A/\mathbb{C} does not satisfy the PBW-property. The ring A is regular hence the Lie-Rinehart algebra $\text{Der}_{\mathbb{C}}(A)$ is a projective A -module of rank two. Hence the canonical map

$$\rho : \text{Sym}_A^*(\text{Der}_{\mathbb{C}}(A)) \rightarrow \text{Gr}(U(A, \text{Der}_{\mathbb{C}}(A)))$$

cannot be an isomorphism.

In Rinehart's classical paper [13] a general PBW-theorem for Lie-Rinehart algebras is formulated and proved. It says that for any Lie-Rinehart algebra $\alpha : L \rightarrow \text{Der}_R(A)$ which is projective as A -module, it follows the canonical map

$$\rho : \text{Sym}_A^*(L) \rightarrow \text{Gr}(U(A, L))$$

is an isomorphism. This theorem cannot be true in complete generality as this example shows. It is expected that most Lie-Rinehart algebras $\alpha : L \rightarrow \text{Der}_R(A)$ that are finitely generated and projective as A -module satisfy the PBW-property.

Example 3.17. *On the deformation groupoid of a Lie-Rinehart algebra and existing results.*

In the paper [10] I claimed to have proved the following result. Let $\alpha : L \rightarrow \text{Der}_R(A)$ be a Lie-Rinehart algebra where L is a projective and finitely generated A -module. In Theorem 4.14, [10] I claim to prove the cohomology group $H^2(L, A)$ parametrize filtered associative algebras (U, U_i) where the canonical map

$$\rho : \text{Sym}_{U_0}^*(U_1/U_0) \rightarrow \text{Gr}(U, U_i)$$

is an isomorphism. An algebra (U, U_i) satisfying this condition is called an almost commutative PBW-algebra or an algebra of twisted differential operators in the literature. It has been extensively studied in [3], [15] and [17]

Example 3.16 shows that the classification from [10] cannot be correct since this classification implies the surjectivity of the map in 3.16.1. All the results in the paper [10] are true for Lie-Rinehart algebras $\alpha : L \rightarrow \text{Der}_R(A)$ that are projective as A -module and satisfies the PBW-property.

Similar classifications of sheaves of algebras of twisted differential operators are claimed to exist in [3] and [17] and it seems these results are wrong for the same reason.

Let $X = \mathbb{P}_{\mathbb{C}}^2$ be the projective plane and let T_X be the tangent sheaf. It is clear that X is a compact Kaehler manifold. Tortella proves in [17] the existence of an isomorphism of sheaves of graded \mathcal{O}_X -algebras

$$\rho : \text{Sym}_{\mathcal{O}_X}^*(T_X) \cong \text{Gr}(\mathcal{U}(\mathcal{O}_X, T_X))$$

where $\mathcal{U}(\mathcal{O}_X, T_X)$ is a global version of Rinehart's universal enveloping algebra. Let $U = \mathbb{C}^* \times \mathbb{C}^* \subseteq X$. Since ρ is a map of sheaves of \mathcal{O}_X -algebras we get when we restrict ρ to U a map

$$\rho_U : \text{Sym}_{\mathcal{O}_X}^*(T_X)_U \cong \text{Gr}(\mathcal{U}(\mathcal{O}_X, T_X))_U$$

which is the canonical map

$$\rho_U : \text{Sym}_A^*(\text{Der}_{\mathbb{C}}(A)) \rightarrow \text{Gr}(U(A, \text{Der}_{\mathbb{C}}(A)))$$

where $A = \mathbb{C}[x, 1/x, y, 1/y]$ from Example 3.16. Since ρ_U cannot be an isomorphism by Example 3.16 it follows ρ cannot be an isomorphism. Hence Tortella's classification turns out to be wrong. This is because Tortella's result is based on the original theorem of Rinehart from 1963 which is not true in complete generality as stated in Rinehart's paper.

There is ongoing work on the problem of classifying Lie-Rinehart algebras satisfying the PBW-property. It is expected that Rinehart's theorem is true for most Lie-Rinehart algebras $\alpha : L \rightarrow \text{Der}_R(A)$ where L is a finitely generated and projective A -module.

4. SOME FAMILIES OF CLASSES IN THE GROTHENDIECK GROUP OF CONNECTIONS

In this section we use results from Section 2 of this paper to construct non-trivial families of classes in the Grothendieck group $\overline{K}_0(L)$ of L -connections, where L is a Lie-Rinehart algebra satisfying the PBW-condition.

Let $\alpha : L \rightarrow \text{Der}_R(A)$ be a Lie-Rinehart algebra satisfying the PBW-condition, where R contains a field of characteristic zero. Let $f \in Z^2(L, A)$ and let $\tilde{V}^{k,i}(A, L(f))$ be the A -module constructed in Definition 2.6. By Theorem 2.10 it follows $\tilde{V}^{k,i}(A, L(f))$ is a finitely generated and projective A -module for any $f \in Z^2(L, A)$. The rank of $\tilde{V}^{k,i}(A, L(f))$ is independent of choice of f .

Let $r(k, i) = rk(\tilde{V}^{k,i}(A, L(f)))$. Let $c = \bar{f} \in H^2(L, A)$ and let f' be another representative for c . It follows by from Section 2 there is a canonical isomorphism

$$\tilde{U}(A, L(f)) \cong \tilde{U}(A, L(f'))$$

of filtered associative rings. There is moreover for any pair of integers $k, i \geq 1$ an isomorphism

$$\tilde{V}^{k,i}(A, L(f)) \cong \tilde{V}^{k,i}(A, L(f'))$$

of left A -modules. Recall the following from Definition 2.8:

Let $c = \bar{f} \in H^2(L, A)$ be a cohomology class where $f \in Z^2(L, A)$ is a representative for c . Let

$$\tilde{V}^{k,i}(c) := \tilde{V}^{k,i}(A, L(f)).$$

Let also

$$\tilde{U}(A, L(c)) := \tilde{U}(A, L(f)).$$

By functoriality it follows $\tilde{V}^{k,i}(c)$ and $\tilde{U}(A, L(c))$ are well defined. The two A -modules $\tilde{V}^{k,i}(A, L(f))$ and $\tilde{V}^{k,i}(A, L(f'))$ are in a canonical way $U^{ua}(L)$ -modules where $U^{ua}(L)$ is the universal algebra of L as defined in [10], Appendix A. hence there are canonical connections

$$\nabla_f : L \rightarrow \text{End}_R(\tilde{V}^{k,i}(A, L(f)))$$

and

$$\nabla_{f'} : L \rightarrow \text{End}_R(\tilde{V}^{k,i}(A, L(f')))$$

Lemma 4.1. *There is a canonical isomorphism*

$$(\tilde{V}^{k,i}(A, L(f)), \nabla_f) \cong (\tilde{V}^{k,i}(A, L(f')), \nabla_{f'})$$

of L -connections.

Proof. Since $f' = f + d^1(v)$ where $v \in Z^1(L, A)$ it follows there is a canonical isomorphism of filtered associative rings

$$\tilde{U}(A, L(f)) \cong \tilde{U}(A, L(f'))$$

Hence there is an isomorphism

$$\tilde{V}^{k,i}(A, L(f)) \cong \tilde{V}^{k,i}(A, L(f'))$$

as $\tilde{U}(A, L(f))$ and $U^{ua}(L)$ -modules. The Lemma follows. \square

Recall the construction of the map $\phi^{k,i}$ from Section 2 in this paper: Let $r(k, i) = rk(\tilde{V}^{k,i}(c))$ and let $\tilde{c} = \frac{1}{r(k,i)}c$. We get from Lemma 4.1 the following: For any

cohomology class $c \in H^2(L, A)$ there is a well defined finitely generated projective A -module $\tilde{V}^{k,i}(\tilde{c})$ and a well defined L -connection

$$\nabla_{\tilde{c}} : L \rightarrow \text{End}_R(\tilde{V}^{k,i}(\tilde{c})).$$

The connection $\nabla_{\tilde{c}}$ is well defined up to isomorphism of L -connections.

We get for all $k, i \geq 1$ a well defined map of sets

$$\psi^{k,i} : H^2(L, A) \rightarrow \overline{K}_0(L)$$

defined by

$$\psi^{k,i}(c) = [\tilde{V}^{k,i}(\tilde{c}), \nabla_{\tilde{c}}].$$

Theorem 4.2. *Assume L is a Lie-Rinehart algebra satisfying the PBW-property, which is finitely generated and projective of rank l as A -module. Let $c, c' \in H^2(L, A)$ and let $k, i, j \geq 1$ be integers. The following holds:*

$$(4.2.1) \quad c_1(\psi^{k,i}(c)) = c \in H^2(L, A)$$

$$(4.2.2) \quad c \neq 0 \Rightarrow \psi^{k,i}(c) \neq 0 \text{ in } \overline{K}_0(L)$$

$$(4.2.3) \quad k \geq 1, i \neq j \geq 1 \Rightarrow \psi^{k,i}(c) \neq \psi^{k,j}(c) \text{ in } \overline{K}_0(L)$$

$$(4.2.4) \quad c \neq c' \Rightarrow \psi^{k,i}(c) \neq \psi^{k,i}(c') \text{ in } \overline{K}_0(L)$$

$$(4.2.5) \quad rk(\tilde{V}^{k,i}(\tilde{c})) = \binom{l+k+i-1}{l} - \binom{l+k-1}{l}$$

Proof. We prove Claim 4.2.1: By definition of the first Chern class we get the following: Let

$$\nabla : L \rightarrow \text{End}_R(V^{k,i}(\tilde{c}))$$

be the connection of curvature type $\frac{1}{r(k,i)}f$ on $\tilde{V}^{k,i}(\tilde{c})$. We get

$$c_1(V^{k,i}(\tilde{c})) = \overline{tr(R_{\nabla})} = \overline{tr(\frac{1}{r(k,i)}f)Id} = \overline{f} = c.$$

Hence Claim 4.2.1 is proved. Claim 4.2.2: This is clear since c_1 is a morphism of groups. Claim 4.2.3: This is clear since when $i \neq j$ it follows the two modules have different rank. Claim 4.2.4: Assume $c \neq c'$ in $H^2(L, A)$ with $\psi^{k,i}(c) = \psi^{k,i}(c')$ it follows

$$c = c_1(\psi^{k,i}(c)) = c_1(\psi^{k,i}(c')) = c'$$

which is a contradiction. Claim 4.2.5 is proved in [10], Lemma 5.2. The Theorem is proved. \square

By Theorem 4.2, Equation 4.2.4 it follows for any $k, i \geq 1$ the map

$$\psi^{k,i} : H^2(L, A) \rightarrow \overline{K}_0(L)$$

is an injection of sets. This gives rise to non-trivial and inequivalent families of classes $\{\psi^{k,i}(c)\}_{c \in H^2(L, A)}$ in $\overline{K}_0(L)$.

Example 4.3. *Non-trivial elements in $\ker(c_1)$*

Assume $c_1, \dots, c_n \in H^2(L, A)$ and $k_0, k_1, \dots, k_n, i_0, i_1, \dots, i_n \geq 1$ are integers. Let

$$\omega = \psi^{k_1, i_1}(c_1) + \dots + \psi^{k_n, i_n}(c_n) - \psi^{k_0, i_0}(c_1 + \dots + c_n) \in \overline{K}(L).$$

Lemma 4.4. *The following holds: $w \neq 0$ in $\overline{K}_0(L)$ in general. Moreover $c_1(w) = 0$.*

Proof. We get from Theorem 4.2 the following:

$$c_1(\omega) = c_1 + \cdots + c_n - (c_1 + \cdots + c_n) = 0$$

since $\psi^{k,i}$ is a section of c_1 for all k, i . One checks that $\omega \neq 0$ in general and the Lemma follows. \square

Let A be a finitely generated regular algebra over the complex numbers and let $X = \text{Spec}(A)$. Assume A/\mathbb{C} satisfies the PBW-property. Let $L = \text{Der}_{\mathbb{C}}(A)$ and assume $\text{rk}(L) = l$ as A -module. It follows $H^2(L, A) = H_{\text{sing}}^2(X, \mathbb{C})$.

We get for any integers $k, i \geq 1$ a map of sets

$$\psi^{k,i} : H_{\text{sing}}^2(X, \mathbb{C}) \rightarrow \overline{K}_0(A)$$

defined by

$$\psi^{k,i}(c) = [\tilde{V}^{k,i}(\tilde{c})].$$

Here $K_0(A)$ is the Grothendieck ring of finitely generated and projective A -modules and $\overline{K}_0(A)$ is the canonical sub ring of $K_0(A)$ of type one from Definition 3.9.

Corollary 4.5. *Assume $H_{\text{sing}}^2(X, \mathbb{C}) \neq 0$. The following holds:*

- (4.5.1) $c_1(\psi^{k,i}(c)) = c \in H_{\text{sing}}^2(X, \mathbb{C})$
- (4.5.2) $c \neq 0 \Rightarrow \psi^{k,i}(c) \neq 0$ in $\overline{K}_0(A)$
- (4.5.3) $k \geq 1, i \neq j \geq 1 \Rightarrow \psi^{k,i}(c) \neq \psi^{k,j}(c)$ in $\overline{K}_0(A)$
- (4.5.4) $c \neq c' \Rightarrow \psi^{k,i}(c) \neq \psi^{k,i}(c')$ in $\overline{K}_0(A)$
- (4.5.5) $\text{rk}(\tilde{V}^{k,i}(\tilde{c})) = \binom{l+k+i-1}{l} - \binom{l+k-1}{l}$

Proof. The Corollary follows from Theorem 4.2. \square

Hence for any pair of integers $k, i \geq 1$ we get a set theoretic map

$$\psi^{k,i} : H_{\text{sing}}^2(X, \mathbb{C}) \rightarrow \overline{K}_0(A).$$

The map $\psi^{k,i}$ is a set theoretic section of the first Chern class map

$$c_1 : \overline{K}_0(A) \rightarrow H_{\text{sing}}^2(X, \mathbb{C}).$$

We get for any pair $k, i \geq 1$ a family of non-trivial classes

$$\{\psi^{k,i}(c)\}_{c \in H_{\text{sing}}^2(X, \mathbb{C})}$$

in $\overline{K}_0(A) \subseteq K_0(A)$. When we vary the integers k, i we get by Corollary 4.5 inequivalent families of classes. Hence the sub-ring $\overline{K}_0(A)$ contains large families of non-trivial classes parametrized by $H_{\text{sing}}^2(X, \mathbb{C})$ which is a finite dimensional complex vector space. The ring $\overline{K}_0(A)$ is not a complex vector space but there is by the previous section a surjection of rings

$$\overline{Ch}_{\mathbb{Q}} : \overline{K}_0(A) \otimes \mathbb{Q} \rightarrow \text{Char}(\text{Der}_{\mathbb{C}}(A))$$

and $\text{Char}(\text{Der}_{\mathbb{C}}(A))$ is isomorphic to $H_{\text{sing}}^2(X, \mathbb{C})$ as vector space over \mathbb{Q} .

Example 4.6. *Real algebraic manifolds.* If A is a finitely generated regular algebra over the real numbers and $\alpha : L \rightarrow \text{Der}_{\mathbb{R}}(A)$ is a Lie-Rinehart algebra satisfying the PBW-property, it follows there is a surjection

$$\overline{K}(L) \otimes \mathbb{Q} \rightarrow H^2(L, A)$$

and $H^2(L, A)$ is a vector space over the real numbers. If $H^2(L, A) \neq 0$ it follows $H^2(L, A)$ has uncountable infinite rank as abelian group. It follows the same holds for $\overline{K}(L) \otimes \mathbb{Q}$. One gets results similar to Corollary 4.5 for $\overline{K}(A)$.

5. ON THE RANK OF THE CHOW GROUP OF AN ALGEBRAIC VARIETY

Let in the following $R \rightarrow A$ be a map of commutative rings with unity where R contains a field of characteristic zero. Let $\alpha : L \rightarrow \text{Der}_R(A)$ be a Lie-Rinehart algebra where L is a finitely generated and projective A -module satisfying the PBW-property. Let the rank of L be l as A -module. Recall the following from Section 2 of this paper: Let $f \in Z^2(L, A)$ be a 2-cocycle and let $k, i \geq 1$ be integers. There is by Theorem 2.10 a finitely generated projective A -module $\tilde{V}^{k,i}(A, L(f))$ and a connection

$$\nabla : L \rightarrow \text{End}_R(\tilde{V}^{k,i}(A, L(f)))$$

with the following properties:

$$(5.0.1) \quad rk(\tilde{V}^{k,i}(A, L(f))) = \binom{l+k+i-1}{l} - \binom{l+k-1}{l}$$

$$(5.0.2) \quad R_{\nabla}(x, y)(v) = f(x, y)v$$

for all $x, y \in L$ and $v \in \tilde{V}^{k,i}(A, L(f))$. The rank of $\tilde{V}^{k,i}(A, L(f))$ is independent of choice of 2-cocycle f . Let $c = \overline{f} \in H^2(L, A)$ be the cohomology class of f . Let $F = \frac{1}{r}f \in Z^2(L, A)$ where $r = rk(\tilde{V}^{k,i}(A, L(f)))$. We get by the results a connection

$$\nabla : L \rightarrow \text{End}_R(\tilde{V}^{k,i}(A, L(F))),$$

and from Theorem 4.2 it follows $c_1(\tilde{V}^{k,i}(A, L(f))) = c \in H^2(L, A)$. Let $\tilde{V}^{k,i}(f) = \tilde{V}^{k,i}(A, L(f))$ and consider $\wedge^d \tilde{V}^{k,i}(f)$ for $1 \leq d \leq r$. It follows $\wedge^d \tilde{V}^{k,i}(f)$ is a finitely generated and projective A -module with a connection

$$\overline{\nabla} : L \rightarrow \text{End}_R(\wedge^d \tilde{V}^{k,i}(f))$$

defined by

$$\overline{\nabla}(x)(v_1 \wedge \cdots \wedge v_d) = \sum_i v_1 \wedge \cdots \wedge \nabla(x)(v_i) \wedge \cdots \wedge v_d.$$

We calculate the curvature of $\overline{\nabla}$:

Proposition 5.1. *The following holds:*

$$R_{\overline{\nabla}}(x, y)(v_1 \wedge \cdots \wedge v_d) = df(x, y)v_1 \wedge \cdots \wedge v_d$$

for $x, y \in L$ and $v_1 \wedge \cdots \wedge v_d \in \wedge^d \tilde{V}^{k,i}(f)$.

Proof. Recall that ∇ has curvature type f . It follows

$$\begin{aligned} R_{\overline{\nabla}}(x, y)(v_1 \wedge \cdots \wedge v_d) &= \\ \sum_i v_1 \wedge \cdots \wedge R_{\nabla}(x, y)(v_i) \wedge \cdots \wedge v_d &= \end{aligned}$$

$$\sum_i v_1 \wedge \cdots \wedge f(x, y)(v_i) \wedge \cdots \wedge v_d =$$

$$f(x, y) \sum_i v_1 \wedge \cdots \wedge v_d = df(x, y)v_1 \wedge \cdots \wedge v_d$$

for all $v_1 \wedge \cdots \wedge v_d \in \wedge^d \tilde{V}^{k,i}(f)$ and $x, y \in L$. \square

We get a generalization of Theorem 5.3 from [10]:

Let $F = \frac{1}{rd}f \in Z^2(L, A)$. The rank of $\wedge^d \tilde{V}^{k,i}(f)$ is independent of 2-cocycle f .

Theorem 5.2. *Let $1 \leq d \leq r = rk(\wedge^d \tilde{V}^{k,i}(f))$. Let $c = \bar{f} \in H^2(L, A)$. There is a connection*

$$\nabla' : L \rightarrow \text{End}_R(\wedge^d \tilde{V}^{k,i}(F))$$

and $c_1(\wedge^d \tilde{V}^{k,i}(F)) = c \in H^2(L, A)$.

Proof. We get the following calculation:

$$R_{\nabla'}(x, y) = dF(x, y)Id_{\wedge^d \tilde{V}^{k,i}(F)} = \frac{d}{rd}f(x, y)Id_{\wedge^d \tilde{V}^{k,i}(F)} =$$

$$\frac{1}{r}f(x, y)Id_{\wedge^d \tilde{V}^{k,i}(F)}.$$

It follows

$$c_1(\wedge^d \tilde{V}^{k,i}(F)) = \text{tr}(R_{\nabla'}(x, y)) =$$

$$\text{tr}\left(\frac{1}{r}f(x, y)Id_{\wedge^d \tilde{V}^{k,i}(F)}\right) = \frac{1}{r}f(x, y)\text{tr}(Id_{\wedge^d \tilde{V}^{k,i}(F)}) =$$

$$\frac{1}{r}f(x, y)r = f(x, y).$$

It follows

$$c_1(\wedge^d \tilde{V}^{k,i}(F)) = \overline{\text{tr}(R_{\nabla'}(x, y))} = \bar{f} = c \in H^2(L, A).$$

The Theorem follows. \square

Example 5.3. *An affine version of Atiyah's conjecture on existence of flat connections.*

Corollary 5.4. *Let $X = \text{Spec}(A)$ where A is a regular algebra of finite type over the complex numbers satisfying the PBW-property. Let $X_{\mathbb{C}}$ be the underlying complex algebraic manifold in the strong topology. Let $c = \bar{f} \in H^2(X_{\mathbb{C}}, \mathbb{C})$ be a non-zero cohomology class. It follows the module $\wedge^d \tilde{V}^{k,i}(F)$ has no flat algebraic connection for any $k, i \geq 1$ and $1 \leq d \leq r$.*

Proof. Assume there is a flat algebraic connection

$$\nabla : L \rightarrow \text{End}_{\mathbb{C}}(\wedge^d \tilde{V}^{k,i}(F)).$$

It follows $c = c_1(\wedge^d \tilde{V}^{k,i}(F)) = \text{tr}(R_{\nabla}) = 0$ which is a contradiction. The Corollary is proved. \square

I believe Corollary 5.4 is well known but include it for lack of a good reference. This conjecture was stated for complex projective manifolds in Atiyah's paper [1]. It is also mentioned in the introduction of Bloch and Esnault's paper [4] as an open problem.

The construction of the algebraic vector bundle $\wedge^d \tilde{V}^{k,i}(F)$ is done using a filtration in the generalized enveloping algebra $\tilde{U}(A, L(f))$ and PBW-theorem for $\tilde{U}(A, L(f))$. The constructions from [10] and this paper are functorial in the Lie-Rinehart algebra L hence they globalize to give results for sheaves of Lie-Rinehart algebras on complex projective manifolds. It might be a globalization of the construction this paper and in [10] may be useful in the study of Atiyah's conjecture. This is work in progress.

Let $L^{k,i}(F) = \wedge^r \tilde{V}^{k,i}(F)$. It follows $L^{k,i}(F)$ is an invertible A -module with a connection

$$\tilde{\nabla} : L \rightarrow \text{End}_R(L^{k,i}(F)).$$

Let $\text{Pic}^L(A)$ be the group of invertible A -modules with an L -connection.

Corollary 5.5. *Assume L is a finitely generated and projective as A -module satisfying the PBW-property. Assume A contains a field of characteristic zero. It follows the Chern class map*

$$c_1 : \text{Pic}^L(A) \rightarrow H^2(L, A)$$

is a surjective map of abelian groups.

Proof. By Theorem 5.2 it follows $c_1(L^{k,i}(F)) = c \in H^2(L, A)$ where $c \in H^2(L, A)$ is an arbitrary class. The Corollary follows. \square

Example 5.6. *Note: For an algebraic curve E over a field K and an open affine subset $U = \text{Spec}(A) \subseteq E$ it follows $L = \text{Der}_K(A)$ is an invertible A -module, hence $H^2(L, A) = 0$. Hence we do not get an estimate on the rank of $\text{Pic}(E)$ using the methods introduced in this section.*

Example 5.7. *Non-trivial elements in $\ker(\text{Ch})$ and the Beauville conjecture.*

Assume in the following $\alpha : L \rightarrow \text{Der}_R(A)$ is a Lie-Rinehart algebra satisfying the PBW-property. Assume R contains a field of characteristic zero. Recall the Chern character

$$\text{Ch} : K_0(L) \rightarrow H^*(L, A).$$

Let $F_1, \dots, F_n \in H^2(L, A)$ and $k_0, k_1, \dots, k_n, i_0, i_1, \dots, i_n \geq 1$ be integers. Let

$$\omega = L^{k_1, i_1}(F_1) \otimes \dots \otimes L^{k_n, i_n}(F_n) \otimes L^{k_0, i_0}(-F_1 - \dots - F_n) \in \text{Pic}^L(A).$$

Let

$$\eta = L^{k_1, i_1}(F_1) + \dots + L^{k_n, i_n}(F_n) + L^{k_0, i_0}(-F_1 - \dots - F_n) \in K_0(L).$$

Proposition 5.8. *The element $\eta - 1$ is non-trivial in general. Moreover $\text{Ch}(\eta - 1) = 0$.*

Proof. From Proposition 3.13 we get the following: Assume $\{E, \nabla\}$ is an L -connection of curvature type f with $f \in Z^2(L, A)$. It follows

$$\text{Ch}([E, \nabla]) = 1 - rk(E) + rk(E) \exp(c_1(E)).$$

We get

$$\text{Ch}(\eta - 1) = 1 - rk(\eta) + rk(\eta) \exp(c_1(\eta)) - 1 = 1 - 1 = 0.$$

One checks the element η is non-trivial in general since the classes $F_i \in H^2(L, A)$ are arbitrary. The Proposition is proved. \square

Hence when we vary the classes F_i and integers k, i we get a large class of non-trivial elements in the kernel of the Chern character.

The element $\omega \in \text{Pic}^L(A)$ is non-trivial in general. Moreover $c_1(\omega) = 0$. This example indicates that the cycle map

$$c_{X, \mathbb{C}} : \text{CH}^*(X) \rightarrow H_{\text{sing}}^*(X, \mathbb{C})$$

where X is a complex affine algebraic manifold is seldom injective on the subring of $\text{CH}^*(X)$ generated by divisors.

The Beauville conjecture says that for a projective hyper-Kähler manifold X , the cycle class map

$$c_{X, \mathbb{C}} : \text{CH}^*(X) \rightarrow H_{\text{sing}}^*(X, \mathbb{C})$$

is injective on the sub-algebra of $\text{CH}^*(X)$ generated by divisors. The invertible module ω globalize to the projective algebraic setting since all constructions in [10] are functorial. It may be a globalization of the invertible module ω can be useful in the study of the Beauville conjecture (see [2]). The Beauville conjecture is implied by the Bloch-Beilinson conjecture which predicts the existence of a functorial filtration $\{F^i\}_{i \geq 0}$ on the Chow group $\text{CH}^*(X)_{\mathbb{Q}}$ where X is a smooth projective variety over a field of characteristic zero. Hence a globalization of the construction in this Example may have applications to the Bloch-Beilinson conjecture on algebraic cycles.

Let R be the field of complex numbers and let A be a finitely generated regular algebra over R where A/R satisfies the PBW-property. Let $L = \text{Der}_R(A)$. It follows L is a finitely generated and projective A -module. Let $X = \text{Spec}(A)$ and let $X_{\mathbb{C}}$ be the underlying complex manifold of X in the strong topology. It follows from [7] there is an isomorphism

$$H_{\text{sing}}^2(X_{\mathbb{C}}, \mathbb{C}) \cong H^2(L, A)$$

where $H_{\text{sing}}^2(X_{\mathbb{C}}, \mathbb{C})$ is singular cohomology of $X_{\mathbb{C}}$ with complex coefficients.

Corollary 5.9. *The first Chern class map*

$$c_1 : \text{Pic}(A) \rightarrow H_{\text{sing}}^2(X_{\mathbb{C}}, \mathbb{C})$$

is a surjective map of abelian groups.

Proof. By Corollary 5.5 there is for any class $c = \bar{f} \in H_{\text{sing}}^2(X_{\mathbb{C}}, \mathbb{C})$ a connection

$$\nabla : L \rightarrow \text{End}_{\mathbb{C}}(L^{k,i}(F))$$

with $c_1(L^{k,i}(F)) = \bar{f} = c \in H_{\text{sing}}^2(X_{\mathbb{C}}, \mathbb{C})$. This proves the Corollary. \square

Since c_1 is surjective, it follows $\text{Pic}(A)$ must be uncountably infinitely generated as abelian group in the case when $H_{\text{sing}}^2(X_{\mathbb{C}}, \mathbb{C}) \neq 0$.

There is a decomposition

$$\text{Pic}(A) = T \oplus (\oplus_{i \in I} \mathbb{Z}e_i)$$

where $I \subseteq \text{Pic}(A)$ is the subgroups of torsion invertible sheaves and $\oplus_{i \in I} \mathbb{Z}e_i$ is a free abelian group on the set I .

Lemma 5.10. *The following holds: $c_1(U) = 0$ for all torsion invertible A -modules $U \in T$.*

Proof. By Lemma 3.6 the following holds. Let $U \in T$ be a torsion invertible A -module. It follows $U^{\otimes n} = 0$ for some positive integer n . We get

$$c_1(0) = c_1(U^n) = nc_1(U) = 0$$

hence $c_1(U) = 0$ since $H_{\text{sing}}^2(X_{\mathbb{C}}, \mathbb{C})$ is a complex vector space. \square

Theorem 5.11. *Assume $H_{\text{sing}}^2(X_{\mathbb{C}}, \mathbb{C}) \neq 0$. It follows $\text{Pic}(A) \otimes k$ has uncountable dimension over k for $k = \mathbb{Q}$ or $k = \mathbb{F}_l$ for any prime l .*

Proof. There is a decomposition $\text{Pic}(A) = T \oplus (\oplus_{i \in I} \mathbb{Z}e_i)$ where T is the torsion invertible A -modules and $\oplus_{i \in I} \mathbb{Z}e_i$ is a free abelian group on the set I . It follows from Lemma 5.10 that $c_1(T) = 0$ hence we get a surjective map

$$c_1 : \oplus_{i \in I} \mathbb{Z}e_i \rightarrow H_{\text{sing}}^2(X_{\mathbb{C}}, \mathbb{C})$$

of abelian groups. This implies I is an uncountable set and the Theorem follows since

$$\text{Pic}(A) \otimes k \cong T \otimes k \oplus (\oplus_{i \in I} ke_i).$$

\square

Corollary 5.12. *Let X be an algebraic variety of finite type over the complex numbers. Assume $U = \text{Spec}(A)$ is an open affine subscheme of X where A/\mathbb{C} satisfies the PBW-property and $H_{\text{sing}}^2(U, \mathbb{C}) \neq 0$. It follows $\text{Pic}(X) \otimes k$ has uncountable infinite dimension over k where k is the field of rational numbers or \mathbb{F}_l the field with l elements where l is any prime.*

Proof. Let $Z = X - U$. There is an exact sequence of abelian groups

$$\text{Pic}(Z) \rightarrow \text{Pic}(X) \rightarrow \text{Pic}(U) \rightarrow 0.$$

The result follows from Theorem 5.11 since the sequence remains right exact when we tensor with k . \square

Example 5.13. *The Neron-Severi group $NS(X)$.*

Assume $X = \text{Spec}(A)$ where A is a regular algebra of finite type over the complex numbers satisfying the PBW-property and $H_{\text{sing}}^2(X_{\mathbb{C}}, \mathbb{C}) \neq 0$. There is an exact sequence of groups

$$0 \rightarrow \text{Pic}_X^0 \rightarrow \text{Pic}_X \rightarrow NS(X) \rightarrow 0$$

where Pic_X is the Picard scheme of X and Pic_X^0 is the connected component of the identity in Pic_X . The group $NS(X)$ is the Neron-Severi group of X . The group $NS(X)$ is a finitely generated abelian group. One may construct $NS(X)$ as follows:

$$NS(X) = Z_1(X) / \text{Alg}_1(X)$$

(see [5], 19.3.1) where $\text{Alg}_1(X)$ means cycles algebraically equivalent to zero. There is an equality

$$\text{Pic}(X) \cong Z_1(X) / \text{Rat}_1(X)$$

where $\text{Rat}_1(X)$ means cycles rationally equivalent to zero. Hence Theorem 5.11 indicates that there is a strict inclusion

$$\text{Rat}_1(X) \subsetneq \text{Alg}_1(X)$$

and a surjective map

$$\text{Pic}(X) \rightarrow NS(X)$$

of abelian groups which is seldom an isomorphism. The group of algebraic cycles $\mathrm{CH}^*(X)$ modulo rational equivalence and $\mathrm{Pic}(X)$ was conjectured (by Severi) to be finitely generated abelian groups. A counterexample to this claim was first given by Mumford in [12].

Corollary 5.14. *Assume $X = \mathrm{Spec}(A)$ where A is a regular algebra of finite type over the complex numbers. Assume A/\mathbb{C} satisfies the PBW-property. If $H_{\mathrm{sing}}^2(X_{\mathbb{C}}, \mathbb{C}) \neq 0$ it follows $\mathrm{CH}^*(X) \otimes k$ has uncountable dimension over k where $k = \mathbb{Q}$ or $k = \mathbb{F}_l$ where l is any prime.*

Proof. The Corollary follows from Theorem 5.11 since there is a split injection $\mathrm{Pic}(A) \subseteq \mathrm{CH}^*(X)$ of abelian groups. \square

Example 5.15. *Algebraic cycles modulo linear equivalence.*

Corollary 5.16. *Assume A is a regular commutative algebra of finite type over the complex numbers such that $X = \mathrm{Spec}(A)$ is irreducible. Assume A/\mathbb{C} satisfies the PBW-property and $H_{\mathrm{sing}}^2(X_{\mathbb{C}}, \mathbb{C}) \neq 0$. It follows $\mathrm{Cl}(X) \otimes k$ is of uncountable infinite dimension over k when $k = \mathbb{Q}$ or $k = \mathbb{F}_l$ where l is any prime.*

Proof. Since X is irreducible and smooth it follows X is integral, hence by [8], Proposition II.6.15 it follows $\mathrm{Pic}(A) \cong \mathrm{CaCl}(X)$ where $\mathrm{CaCl}(X)$ is the group of Cartier divisors on X modulo principal divisors. Since X is smooth it follows by [9] Theorem 20.3 that X is locally factorial. It follows $\mathrm{CaCl}(X) \cong \mathrm{CL}(X)$. Hence $\mathrm{CaCl}(X) \otimes k \cong \mathrm{CL}(X) \otimes k$. The Corollary follows from Theorem 5.11. \square

Corollary 5.17. *Let X be an smooth algebraic variety of finite type over the complex numbers. Assume $U = \mathrm{Spec}(A) \subseteq X$ is an irreducible open affine subset where A/\mathbb{C} satisfies the PBW-property. Assume $H_{\mathrm{sing}}^2(U, \mathbb{C}) \neq 0$. It follows $\mathrm{Cl}(X) \otimes k$ has uncountable infinite dimension over k where $k = \mathbb{Q}$ or $k = \mathbb{F}_l$ where l is any prime.*

Proof. Since X is smooth it follows X is regular in codimension one and locally factorial, hence there is by [8] Proposition II.6.5 a surjection of abelian groups

$$\rho : \mathrm{Cl}(X) \rightarrow \mathrm{Cl}(U).$$

The result now follows from Corollary 5.16 since $\rho \otimes k$ is a surjective map. \square

Hence it appears $\mathrm{Cl}(X)$ is seldom finitely generated.

Let in the following X be a smooth complex algebraic variety of finite type over \mathbb{C} . Let $\mathrm{CH}^*(X)$ denote the Chow-group of X : Algebraic cycles on X modulo rational equivalence of algebraic cycles. Since X is smooth it follows the intersection product gives $\mathrm{CH}^*(X)$ a structure of commutative ring (see [5]).

Corollary 5.18. *Let X be a smooth algebraic variety of finite type over the complex numbers with an open affine subset $U = \mathrm{Spec}(A) \subseteq X$ where A/\mathbb{C} satisfies the PBW-property. Assume $H_{\mathrm{sing}}^2(U, \mathbb{C}) \neq 0$. It follows $\mathrm{CH}^*(X) \otimes k$ has uncountable infinite dimension over k when $k = \mathbb{Q}$ or $k = \mathbb{F}_l$ for any prime l .*

Proof. Let $Z = X - U$ be the complement of U . By Section 1.8 in [5], Proposition 1.8 there is a localization sequence for Chow groups

$$\mathrm{CH}^*(Z) \rightarrow \mathrm{CH}^*(X) \rightarrow \mathrm{CH}^*(U) \rightarrow 0.$$

By Theorem 5.11 it follows $\mathrm{CH}^*(U) \otimes k$ has uncountable infinite dimension for $k = \mathbb{Q}$ or $k = \mathbb{F}_l$ for any prime l . Since the sequence is right exact this implies the same holds for $\mathrm{CH}^*(X) \otimes k$. \square

A similar result holds for the Grothendieck group $K_0(X)$ of finite rank algebraic vector bundles on X , since for any open affine subset $U \subseteq X$ it follows there is an exact sequence of abelian groups

$$K_0(Z) \rightarrow K_0(X) \rightarrow K_0(U) \rightarrow 0$$

where $Z = X - U$.

Acknowledgements. The author wants to thank Marat Rovinsky and an anonymous referee for communicating Example 3.16. Thanks also to Pierre Deligne and Claire Voisin for answering questions on Chow groups and Hodge theory.

REFERENCES

- [1] M. F. Atiyah, Complex analytic connections in fiber bundles, *Trans. of the AMS* no. 85 , 181-207 (1957)
- [2] A. Beauville, On the splitting of the Bloch-Beilinson filtration, in *Algebraic cycles and motives (vol. 2)* London Math. Soc. Lecture Notes 344 *Cambridge University Press* (2007)
- [3] A. Beilinson, J. Bernstein, A proof of the Jantzen conjectures, *Advances in Soviet Math.* vol. 16 (1993)
- [4] S. Bloch, H. Esnault, Algebraic Chern Simons theory, *Am. J. of Math.* 119, no. 4, 903-952 (1997)
- [5] W. Fulton, Intersection theory, *Springer Verlag* (1998)
- [6] T. Geisser, Parshin's conjecture revisited, *EMS Series of Congress Reports* 413-425 (2008)
- [7] A. Grothendieck, On the De Rham cohomology of algebraic varieties, *Publ. Math. IHES Sci.* no. 29, 95-103 (1966)
- [8] R. Hartshorne, Algebraic geometry, *Springer Verlag*, GTM. no. 52 (1983)
- [9] H. Matsumura, Commutative ring theory, Cambridge Studies in Advanced Mathematics, *Cambridge University Press* no. 336 (1989)
- [10] H. Maakestad, Algebraic connections on projective modules with prescribed curvature, *J. of Algebra* no. 436, 161-227 (2015)
- [11] H. Maakestad, The Chern-character for Lie-Rinehart algebras, *Ann. Inst. Fourier* no. 55, 2551-2574 (2005)
- [12] D. Mumford, Rational equivalence of 0-cycles on surfaces, *J. Math. Kyoto Univ.* no. 9, 195-204 (1969)
- [13] G. S. Rinehart, Differential forms on general commutative algebras, *Trans. Am. Math. Soc.* no. 108 (1963)
- [14] C. Schoen, Complex varieties for which the Chow group mod n is not finite, *J. Alg. Geom.* no. 11, 41-100 (2002)
- [15] C. I. Simpson, Moduli of representations of the fundamental group of a smooth projective variety I, *Publ. Math. IHES* no. 79 (1994)
- [16] J. Tate, Conjectures on algebraic cycles, *American Mathematical Society. Proc. Symp. Pure Math.*, no. 55, 71-83 (1994)
- [17] P. Tortella, Γ -modules and holomorphic Lie algebroid connections, *Centr. Eur. J. Math.* no. 10 (2012)
- [18] B. Totaro, Complex varieties with infinite Chow groups modulo 2, *arXiv:1502.0207* (2015)
E-mail address: h_maakestad@hotmail.com